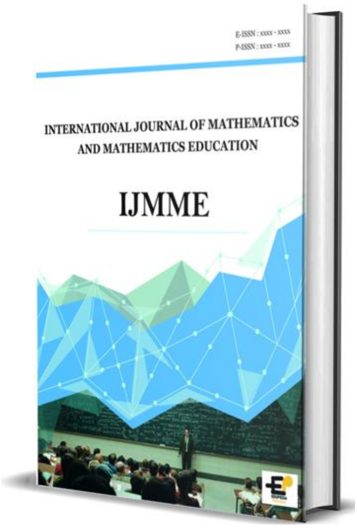

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Small Pseudo Principally Quasi-Injective Acts

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Abstract

This work aims to present a small pseudo principally quasi-injective acts as a novel generalization of pseudo principally quasi-injective S-acts over monoids. If each S-monomorphism from a small principal subact of an S-act M_S into N_S can be extended to S-homomorphism from M_S into N_S, an S-act N_S is termed as a small pseudo principally M-injective. If an S-act M_S is a small pseudo principally M-injective, it is called a small pseudo principally quasi-injective. This type of generalization has several properties. Additionally, the circumstances under which subacts inherit the Small Pseudo, principally quasi-injective acts, are studied. Examples are provided to demonstrate this concept. Small pseudo principally quasi-injective acts can coincide with small principally quasi-injective acts if certain criteria are met. We discuss new characterizations of small principally quasi-injective acts. The link between classes of small, principally quasi-injective acts with other classes of injectivity is shown.

Introduction

An S-act M under the action of S is a non-empty set equipped with a function $f: M \times S \longrightarrow M$ such that f(m,s)↦ms and it is denoted by M_S. The following properties hold for all m∈ M and s, t∈S: (1) m1=m (2)m(st)=(ms)t (3)m0=Θ. Here, 0,1 is the zero and identity element of S and M, respectively. Kilp et al. (2000) defines a sub-act as follows: a sub-act N of M_S is a non-empty subset such that xs ∈ N satisfies for all x∈N and s∈S. The concept of an S-act is also known by other terminologies, such as S-system, S-sets, S-operands, S-polygons, and S-automata (Kilp et al., 2000). We refer the reader to the following references for more details about S-acts and injective acts (Abdul-Kareem, 2020; Yan, 2011).

Consider two S-acts A_S and B_S. A mapping $g: A_S \longrightarrow B_S$ is called S-homomorphism if g(as)=g(a)s for all a∈AS and s∈S (Lopez ,1976). An S-congruence ρ on a right S-act MS is an equivalence relation on MS such that whenever (a,b) ∈ ρ, then (as, bs) ∈ρ for all s ∈ S. The identity S-congruence on MS will be denoted by I_M such that (a,b)∈ I_M if and only if a=b (Hinkle,1973). The congruence psi sub-M is called singular on MS, and it is defined by $a\psi_M b$ if and only if ax = bx for all x in some ∩-large right ideal of S (Lopez,1979). For S-act MS, $H \subset S$, $K \subset M \times M$, $T \subset M$, $J \subset S \times S$:(1)$\ell_M(H) = \{m, n \in M \times M \mid mx =$

nx for all $x \in H\}$(2)$\gamma_S(K) = \{s \in S \mid as = bs$, for all $a, b \in K\}$(3)$\gamma_S(T) = \{(a,b) \in S \times S \mid ta = tb$ for all $t \in T\}$(4)$\ell_M(J) = \{a \in M | am = an$ for all $m, n \in J\}$ (Jupil,2008).

If one element generates an S-act AS, then it is called a principal act, and it is denoted by$A_S = < u >$, whereu $\in$ A,then $A_S = uS$(Kilp et al., 2000, P.63). An S-act BS is a retract of an S-act AS if and only if there exists a subact N of AS and epimorphism $f: A_S \longrightarrow Ns$ such that $B_S \cong N$ and f(n)=n for everyn $\in$ N (Kilp et al., 2000). Let M_S HS be the right S-acts. An S-act E is called injective if for every S-monomorphism$\mathbf{f}: \mathbf{M_S} \longrightarrow \mathbf{H_S}$ and every S-homomorphism $\mathbf{g}: \mathbf{M_S} \longrightarrow \mathbf{E}$, there is an S-homomorphism $\mathbf{h}: \mathbf{H_S} \longrightarrow \mathbf{E}$ such that hf = g (Berthiaume,1967). A right S-acts K_S is called an M-injective if for each S-monomorphism f from S-act B_S into S-act M_S and every homomorphism$\mathbf{g}: \mathbf{B_S} \longrightarrow \mathbf{K_S}$,there is S-homomorphism$\mathbf{h}: \mathbf{M_S} \longrightarrow \mathbf{K_S}$, such that hf=g. Thus, K_S is injective if and only if K_S is M-injective for all S-act M_S (Yan et al., 2007). Berthiaume (1967) studied injective S-acts. Then, injectivity on S-acts is generalized to quasi-injectivity, such that an S-act K_S is quasi-injective if K_S is K-injective (Lopez,1979). Then, the author presented a generalization of quasi-injective acts (Shaymaa, 2015), principally quasi-injective and quasi-injective acts. Besides, the author introduced a generalization of principally quasi-injective acts, which was a small principally quasi-injective act. A small sub-act N of a right S-act M_S is called small (or superfluous) in M_S if for every sub-act H of M_S, N∪H=M_S implies H=M_S. Let M_S be a right S-act. A right S-act K_S is called a small principally M-injective (simply SP-M-injective) if every S-homomorphism from a small and principal sub-act of M_S to K_S can be extended to an S-homomorphism from M_S to K_S. A right S-act M_S is called a small principally quasi-injective (simply Small PQ-injective) if it is SP-M-injective (Abdul-Kareem & Ahmed, 2022). Furthermore, the author introduced another generalization of principally quasi-injective, a pseudo principally quasi-injective act. An S-act H_S is called a pseudo principally M-injective (for short pseudo-PM-injective) if each S-monomorphism from a principal subject of an S-act M_S into H_S can be extended to S-homomorphism from M_S into H_S. An S-act M_S is called pseudo principally quasi-injective if it is pseudo principally M-injective (if this is the case, we write M_S is pseudo-PQ-injective) (Abbas & Shaymaa, 2015).

Wongwai and Sthityanak (2012) introduced a generalization for the small principally quasi-injective module, namely the tiny pseudo principally quasi-injective module. This motivated us to extend this notion to S-acts and obtain interesting results. Throughout this paper, the basic S-act is a unitary right S-act with zero, consisting of a zero monoid.

Materials and Methods

This section is divided into two parts: The first part introduces and explores a novel generalization of pseudo principally quasi-injective S-acts, referred to as small pseudo principally quasi-injective S-acts. Also, we answer the question of when sub-acts inherit the property of small pseudo, principally quasi-injective. Besides, the characterizations of this new class of acts were illustrated, for example (remarks and examples (2.1.2) (3), lemma (2.1.3), and proposition (2.1.4)). Additionally, an example is given to clarify this notion, like (remarks and examples (2.1.2) (2)). Some known results on small pseudo principally quasi-injective for general modules are generalized to S-acts. In the second part, we examine the relationships between small pseudo principally quasi-injective S-acts and other injectivity classes, such as small PQ-injectives. We also identify conditions under which pseudo-PQ-injective S-acts coincide with PQ-injective and pseudo-QP-injective S-acts.

Result and Discussion

Small Pseudo Principally Quasi-Injective Acts:

Definition (2.1.1): An S-act N_S is called a small pseudo principally M-injective (for short SPPM-injective) if each S-monomorphism from small principal subact of an S-act M_S into N_S can be extended to S-homomorphism from M_S into N_S. An S-act M_S is called a small pseudo principally quasi-injective if it is a small pseudo principally M-injective (In this case, we write MS as a small pseudo-PQ-injective).

Remarks and Example (2.1.2):

1. Every small principally quasi-injective (and hence quasi-injective) act is a small pseudo-PQ-injective. But the converse is not valid in general, but we do not have an example yet.

2. Let M_S act where $S = \begin{pmatrix} X & X \\ 0 & X \end{pmatrix}$ and X is a field. Let $A_S = \begin{pmatrix} X & X \\ 0 & 0 \end{pmatrix}$. Then, A_S is the SPP-M-injective act.

Proof: It is easy to show that $B = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is the only nonzero small and principal subject of M_S. Let $\alpha\colon B \longrightarrow A_S$ be S-monomorphism. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in B$, so there exists a, b∈ X such that $\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Then $\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \alpha\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. It implies that a=0. Define $\overline{\alpha}\colon M_S \longrightarrow A_S$ by $\overline{\alpha}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} aa_1 & bb_1 \\ 0 & 0 \end{pmatrix}$ for every a, b, c ∈ X. It is clear that $\overline{\alpha}$ is an S-homomorphism and $\overline{\alpha}$ is an extension of α . Thus, A_S is an SPP-M-injective act.

Now, we give a characterization of a small pseudo-PQ-injective act

3. Retract subact of small pseudo principally quasi-injective act (small pseudo-PQ-injective) is small pseudo principally M-injective (SPPM-injective).

Proof: Let M_S be a small pseudo-PQ-injective act and N be a retract cyclic subact of M_S. Let A be a small and principal subact of M_S with $f : A \longrightarrow N$ be S-monomorphism. Define $\alpha(= j_N o f) : A \longrightarrow M_S$, where j_N is the injection map of N into M_S, so α is S-monomorphism. Since M_S is a small pseudo-PQ-injective act, so there exists S-homomorphism $\beta : M_S \longrightarrow M_S$ such that $\beta o i_A = \alpha$, where $i_A$be the inclusion map of A into M_S. Now, let π_N be the projection map of M_S onto N. Then, define $\sigma(= \pi_N\ \beta) : M_S \longrightarrow N$. Thus, we have that $\sigma o i_A = \pi_N \circ \beta \circ i_A = \pi_N \circ \alpha = \pi_N \circ j_N \circ f = f$. Therefore, an S-homomorphism σ is extends f and N is SPPM-injective act.

In the next lemma (2.1.3) and proposition (2.1.4), another characterization of a small pseudo-PQ-injective act will be illustrated.

Lemma (2.1.3): Let N be a small subact of S-act MS. If N is the SPPM-injective subact of M_S, then N is a retract of M_S.

Proof: Let α be S-monomorphism from small principal subact N of S-act M_S into M_S and I_N be the identity map of N. Then, the SPPM-injectivity of N implies that there exists S-homomorphism

$g: M_S \longrightarrow N$ such that $I_N = g \circ \alpha$, hence α is a retraction. Therefore $N \cong \alpha(N)$ is a retract of M_s.

Proposition (2.1.4): Let M_S be S-act. If N_S is SPPM-injective, then N_S is an SPPA-injective act for any principal subact A of M_S.

Proof: Let X be the small principal subact of principal subact A of MS, then X small in MS by proposition (2.2.4) in the article of the author entitled small principally quasi-injective acts (Abdul-Kareem and Ahmed, 2022), and let f be any S-monomorphism of X into S- act N_S. Let $i_x(i_A)$ be the inclusion map of X(A) into A (M_S) respectively. Since N_S is SPPM-injective, then there exists S-homomorphism $g : M_S \longrightarrow N_S$ such that $g \circ i_A \circ i_X = f$. Define S-homomorphism h by $h(= g \circ i_A) : A \longrightarrow N$, then $\forall x \in A$ we have $h(x) = h(i_X(x)) = (g \circ i_A)(i_X(x)) = (g \circ i_A \circ i_X)(x) = f(x)$, which implies that h extends f and N_S is SPPA- an injective act.

Theorem (2.1.5): Let M_1 and M_2 be two S-acts. If $M_1 \oplus M_2$ is small pseudo-PQ-injective act, then M_1 and M_2 are mutually SP-injective.

Proof: Let $M_1 \oplus M_2$ be SPPQ-injective act. Let A be a small principal subact of M_2, and f an S-homomorphism from into M_1. let j_1 and π_1 be the injection and projection maps of M_1 into $M_1 \oplus M_2$ and $M_1 \oplus M_2$ onto M_1. Define $\alpha : A \longrightarrow M_1 \oplus M_2$ by $\alpha(a) = (f(a), a)$, $\forall a \in A$. It is easy to check that α is S-monomorphism. Since $M_1 \oplus M_2$ is small pseudo-PQ-injective act, so by proposition (2.1.4), $M_1 \oplus M_2$ is $SPPM_2$-injective. Hence, there exists S-homomorphism g from M_2 into $M_1 \oplus M_2$ such that $g \circ i = \alpha$. Now, put $h(= \pi_1 \circ g) : M_2 \longrightarrow M_1$ and figure (1) explain that:

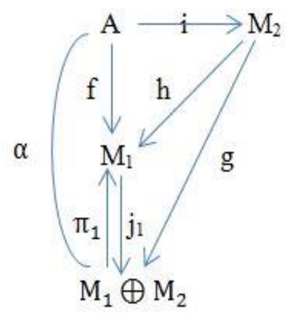

Figure 1. Illustrate that $M_1 \oplus M_2$ is SPPM2-injective act

Thus,$\forall a \in A$, we have $h \circ i(a) = \pi_1 \circ g \circ i(a) = \pi_1 \circ \alpha(a) = \pi_1 (\alpha(a)) = \pi_1(f(a), a) = f(a)$. This proves that M_1 is SPM_2-injective S-act.

Corollary (2.1.6): Let $\{M_i\}_{i\in I}$ be a family of S-acts, where I is a finite index set. If $\oplus_{i\in I} M_i$ is a small pseudo-PQ-injective act, then M_j is the SPM_K-injective act for all $j, k \in I$.

Lemma (2.1.7): Let $\{N_i\}_{i\in I}$ be a family of S-acts, where I is a finite index set. Then, the direct product $\prod_{i\in I} N_i$ SPM-injective if and only if N_i is SPM-injective for every $i \in I$.

Proof:⟹)assume that $N_S = \prod_{i\in I} N_i$ is SPM-injective S-act. Let X be a small and principal subact of M_S, f an S- homomorphism of X into N_i, and φ_i , π_i be the injection and projection map of N_i into N_S and N_S onto N_i, respectively. Since N_S is SPM-injective, so there exists an S-homomorphism $g : M_S \longrightarrow N_S$ such that $g \circ i = \varphi_i \circ f$, where i is the inclusion map of X into M_S. Then, define $h(= \pi_i \circ g) : M_S \longrightarrow N_i$ such that $h \circ i = \pi_i \circ g \circ i = \pi_i \circ \varphi_i \circ f = f$. Thus, N_i is SPM-injective S- act.

⟸) Assume that N_i is SPM-injective for each $i \in I$. Let X be a small and principal subact of M_S, f be an S-homomorphism of X into N_S, and φ_i , π_i be the injection and projection maps of N_i into N_S and N_S onto N_i, respectively. Since N_i is an SPM-injective S- act, so there exists S-homomorphism $\beta_i : M_S \longrightarrow N_i$ such that $\beta_i \circ i = \pi_i \circ f$, where I will be the inclusion map of X into M_S. Now, define an S-homomorphism $\beta(= \varphi_i \circ \beta_i) : M_S \longrightarrow N_S$, then $\beta o i = \varphi_i \circ \beta_i \circ i = \varphi_i \circ \pi_i \circ f = f$. Therefore, N_S is an SPM-injective act.

Corollary (2.1.8): For any integer $n \geq 2$, M^n is a small pseudo-PQ-injective if and only if MS is a Small-PQ-injective act.

Let M_S be S-act. For all elements, $m \in M_S$, with $\alpha \in T$=End (M), define:

$A_m = \{ n \in M_S \mid \gamma_s(n) = \gamma_s(m) \}$;

$S_{(\alpha,m)} = \{ \beta \in T \mid \ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS) \}$;

$B_m = \{\alpha \in T \mid \ker\alpha \cap (mS \times mS) = I_{mS}\}$.

Proposition (2.1.9): Let M_S be an S-act with T=End (M), the following conditions are equivalent for an element $m \in M_S$:

1. M_S is small pseudo principally injective (SPPM-injective),

2. $A_m = B_m \bullet m$,

3. If $A_m = A_n$, then $B_m \bullet m = B_n \bullet n$,

4. For every S-monomorphism $\alpha : mS \longrightarrow M_S$ and $\beta : mS \longrightarrow M_S$, there exists $\sigma \in T$ such that $\alpha = \sigma \circ \beta$

Proof: (1⟶2) Let $n \in A_m$, this implies $A_m = A_n$, hence $\alpha : mS \longrightarrow M_S$ defined by $\alpha(ms) = ns$, $s \in S$. Let $ms_1 = ms_2$, this implies $(s_1 , s_2) \in \gamma_s(m) = \gamma_s(n)$, then $ns_1 = ns_2$. Hence, $\alpha(ms_1) = \alpha(ms_2)$ and α is well-defined and for the reverse steps, we obtain that α is S-monomorphism, so by (1), there exists an S-homomorphism $\beta \in T$ extends α. Then, $\forall\, m \in M_S$, we have $\beta(m) = \alpha(m) = n = \beta \bullet m$, so $\beta \in B_m$ {In fact, if $(ms , mt) \in \ker\beta \cap (mS \times mS)$ then $\beta(ms) = \beta(mt)$ and $ms = mt$. So, $\ker\beta \cap (mS \times mS) = I_{mS}$}. Conversely, if $\beta \bullet m \in B_m \bullet m$, then $\beta \in B_m$, that is $\ker\beta \cap (mS \times mS) = I_{mS}$. It is obvious that $\gamma_S(m) \subseteq \gamma_S(\beta m)$, since for $(r, s) \in \gamma_S(m)$, we have mr = ms, since β is well-defined, so $\beta(mr) = \beta(ms)$. Thus, $\beta(m)r = \beta(m)s$ which implies that $(r, s) \in \gamma_S(\beta m)$. Now, if $\beta(mr) = \beta(ms)$ and $(mr, ms) \in \ker\beta \cap (mS \times mS) = I_{mS}$, then mr =ms and $(r, s) \in \gamma_S(m)$. Hence, $\gamma_S(\beta m) \subseteq \gamma_S(m)$. Then, $\gamma_S(\beta m) = \gamma_S(m)$.Therefore, $\beta m \in A_m$.

(2→3) Let $A_m = A_n$. Then, $A_m = B_m \bullet m$, $A_n = B_n \bullet n$. So, $B_m \bullet m = B_n \bullet n$.

(3→4) Let $\alpha : mS \longrightarrow M_S$, $\beta : mS \longrightarrow M_S$ be S-monomorphisms. Then, $\gamma_S(\beta m) = \gamma_S(\alpha m)$. Since, for $(s,t) \in \gamma_S(\beta m)$ then $\beta(ms) = \beta(mt)$. Since β is monomorphism, so ms = mt. Since α is well-defined, so $\alpha(ms) = \alpha(mt)$. This means $\gamma_S(\beta m) \subseteq \gamma_S(\alpha m)$. In similar way, we can find $\gamma_S(\alpha m) \subseteq \gamma_S(\beta m)$, thus $\gamma_S(\alpha m) = \gamma_S(\beta m)$, which implies $A_{\alpha m} = A_{\beta m}$,then by (3) $B_{\alpha m}\, \alpha m = B_{\beta m} \beta m$. Since $\ker I_M \cap \alpha(mS) \times \alpha(mS) = I_{\alpha(mS)}$, so $1_M \in B_{\alpha m}$. Then $\alpha m \in B_{\beta m}\, \beta m$, so there exists $\sigma \in B_{\beta m}$ such that $\alpha = \sigma\beta$.

(4→1) Let $\beta = i_{mS}$ be the inclusion map of mS.

Proposition (2.1.10): Let M_S be small pseudo principally injective S-act with T = End (M). Then, for $\alpha \in T$, we have: $S_{(\alpha,m)} = B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$, $\forall m \in M_S$.

Proof: Let $\beta \in S_{(\alpha,m)}$, this means $\beta \in T$ and $\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$. We claim that $\gamma_S(\alpha m) = \gamma_S(\beta m)$. In fact, if $(s,t) \in \gamma_S(\alpha m)$, then $\alpha(ms) = \alpha(mt)$ which implies $(ms, mt) \in \ker\alpha \cap (mS \times mS)$ and since $\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$by the proof. So, $(ms, mt) \in \ker\beta \cap (mS \times mS)$ which implies $\beta(ms) = \beta(mt)$ and then $\beta(m)s = \beta(m)t$. Thus $s,t \in \gamma_S(\beta m)$. Hence, $\gamma_S(\alpha m) \subseteq \gamma_S(\beta m)$, similarly we have $\gamma_S(\beta m) \subseteq \gamma_S(\alpha m)$ and then we obtain $\gamma_S(\alpha m) = \gamma_S(\beta m)$. Then, we have $\beta \in A_{\alpha m}$. Since $A_{\alpha m} = B_{\alpha m}\, \alpha m$ by proposition (2.1.9), so $\beta \in B_{\alpha m}\, \alpha m$ and since $\beta(ms) = \beta(mt)$, where $\beta \in T$, thus $\beta \in \ell_T\,(mS \times mS)$ and then, $\beta \in B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$. This means $S_{(\alpha,m)} \subseteq B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$... (1). Conversely, let $\beta \in B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$, so $\beta \in B_{\alpha m}\, \alpha$ or $\beta \in \ell_T\,(mS \times mS)$. If $\beta \in \ell_T\,(mS \times mS)$, so $\beta \in T$ and $\beta(ms) = \beta(mt)$. If $\beta \in B_\alpha\, \alpha$, so there exists $\varphi \in B_\alpha$ such that $\beta = \varphi \circ \alpha$. Also, $\ker\varphi \cap (\alpha(mS) \times \alpha(mS)) = I_{\alpha(mS)}$ and $\ker\beta \cap (\alpha(mS) \times \alpha(mS)) = I_{\alpha(mS)}$. Now, if $(ms, mt) \in \ker\varphi\alpha \cap (mS \times mS)$, then $\varphi\alpha(ms) = \varphi\alpha(mt)$.Hence $(\alpha(ms), \alpha(mt)) \in \ker\varphi \cap (\alpha(mS) \times \alpha(mS)) = I_\alpha$. This implies that $(ms, mt) \in \ker\alpha \cap (mS \times mS)$. Thus, $\ker\beta \cap (mS \times mS) \subseteq \ker\alpha \cap (mS \times mS)$(1). If $(ms, mt) \in \ker\alpha \cap (mS \times mS)$, so $\alpha(ms) = \alpha(mt)$, since $\varphi \in T$ and it is well-defined, so $\varphi\alpha(ms) = \varphi\alpha(mt)$ which implies $\beta(ms) = \beta(mt)$ and then $(ms, mt) \in \ker\beta \cap (mS \times mS)$.Thus,$\ker\alpha \cap (mS \times mS) \subseteq \ker\beta \cap (mS \times mS)$...(2). From (1) and (2), we have $\ker\alpha \cap (mS \times mS) = \ker\beta \cap (mS \times mS)$ and then $\beta \in S_{(\alpha,m)}$.

Proposition (2.1.11): Let M_S be small pseudo principally injective S-act with T = End (M) and $\alpha \in T$ $m \in M_S$. Then: $\alpha \in B_m$ if and only if $B_m = B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$.

Proof:⟹) Let $\alpha \in B_m$ and $f \in S_{(\alpha,m)}$, so $\ker f \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$, but $\ker\alpha \cap (mS \times mS) = i_{mS}$, hence $\ker f \cap (mS \times mS) = i_{mS}$, which implies $f \in B_m$ .Thus , $S_{(\alpha,m)} = B_m$, so by proposition (2.10) $B_m = B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$.

⟸) Assume that $B_m = B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$ and $\alpha \in T$, $\alpha \notin B_m$. Then, we have $\ker\alpha \cap (mS \times mS) \neq I_{mS}$,, so there exists $(ms, mt) \in \ker\alpha \cap (mS \times mS)$ with $ms \neq mt$, then $\alpha(ms) = \alpha(mt)$. Since$1_M \in B_m$, so $\ker I_M \cap (mS \times mS) = I_{mS}$. But, since $S_{(\alpha,m)} = B_m = B_{\alpha m}\, \alpha \cup \ell_T\,(mS \times mS)$, hence $I_M \in S_{(\alpha,m)}$ and then, $\ker\alpha \cap (mS \times mS) = \ker I_M \cap (mS \times mS)$. Thus, $\ker\alpha \cap (mS \times mS) = I_{mS}$ which implies ms=mt and this is a contradiction with $ms \neq mt$. So $\alpha \in B_m$ implies a contradiction.

Recall that Soc_N (M) represent homogeneous component of Soc(M) containing N. Thus, we denote $Soc_N(M) := \cup \{X \text{ be subact of } M_S \mid X \cong N\}$[24].

Proposition (2.1.12): Let M_S be small pseudo principally injective S-act with T= End(M). Then:

1. If N is a simple subact of M_S, then $Soc_N(M) = TN$

2. If nS is a simple S-act, $n \in M_S$. Then, Tn is a simple T-act.

3. $Soc(M_S) = Soc({}_T M)$.

Proof: 1. Let $N_1 \subseteq Soc_N(M_S)$, and $f: N \longrightarrow N_1$ be an isomorphism, where $N_1 \subseteq M_s$. If $N = nS$, then $\gamma_S(n) = \gamma_S(f(n))$. Since, if $(s,t) \in \gamma_S(n)$, then ns=nt, since f is well-defined, so f(ns)=f(nt). This implies f(n)s = f(n)t and $(s,t) \in \gamma_S(f(n))$, so $\gamma_S(n) \subseteq \gamma_S(f(n))$. Conversely, let $(s,t) \in \gamma_S(f(n))$, so f(ns) = f(nt). Since f is monomorphism, so ns = nt. This implies that $(s,t) \in \gamma_S(n)$, so $\gamma_S(f(n)) \subseteq \gamma_S(n)$. Thus $\gamma_S(n) = \gamma_S(f(n))$, which implies $B_n \bullet n = B_{fn} \bullet fn$ by proposition (2.1.9). Thus $fn \in B_n \bullet n \subseteq Tn \subseteq TN$. Hence, if g is an extension of f to T, we have $N_1 = f(nS) = g(nS) \in T$. Thus $Soc_N(M_S) \subseteq TN$. The other inclusion always holds, this means $TN \subseteq Soc_N(M_S)$, since for $\alpha \in TN$, $\alpha: N \longrightarrow N$ be identity map and since $N \cong N$ and N be subact of M_S, so $\alpha(N) = N \subseteq Soc_N(M_S)$ which implies $TN \subseteq Soc_N(M_S)$. Therefore, $Soc_N(M_S) = TN$.

2. Let $\alpha \in T$, $\alpha : M_S \longrightarrow M_S$, since M_S is a small pseudo principally injective, so $\alpha_1 (= \alpha|_{nS}) : nS \longrightarrow M_S$ is S-monomorphism. Since nS is a simple subact of M_S, so $\alpha_1 : nS \longrightarrow \alpha_1(nS)$ is an S-isomorphism. Thus, let $\sigma: \alpha_1(nS) \rightarrow nS$ be its inverse. For $\Theta \neq \alpha n \in Tn$ and if $g \in T$ extends σ, then $g(\alpha_1(n)) = \sigma(\alpha_1(n)) = n \in T\alpha n$. Therefore, $Tn \subseteq T\alpha n$. Then, $Tn = T\alpha n$ whence $T\alpha n \subseteq Tn$, such that if we take $\beta\alpha n \in T\alpha n$ and $\beta \in T$ then, since $\beta \in T$ and $\alpha \in T$, so $\beta\alpha \in T$. Thus, $\beta\alpha n \in Tn$ and $T\alpha n \subseteq Tn$.
3. This is followed by (2).
Recall that an S-homomorphism f, which maps an S-act M_S into an S-act N_S is said to be split if there exists S-homomorphism g, which maps N_S into M_S such that $fg = 1_N$ (Hinkle,1973).

Proposition (2.1.13): Let M_S be small pseudo principally injective S-act with T= End (M). Then:

1. If N and K are isomorphic small principal subact of M_S and K is a retract of M_S, then N is also a retract of M_S.

2. Every small pseudo principally injective has C_2 –condition

Proof: It is obvious that (1) implies (2), so it is enough to prove (1). Let N be a subact of M_S and i be the inclusion map of N into M_S. It is enough to prove that inclusion map split. Let $\alpha: N \longrightarrow K$ be an S-isomorphism. Since K is a retract of M_S, so there exists S-homomorphims $\pi : M_S \longrightarrow K$ and $j: K \longrightarrow M_S$ projection and injection map respectively. Let i_1 be the inclusion map of N into M_S and α^{-1} be the inverse map of α (since α is S-isomorphism). Since M_S is small pseudo principally injective, so there exists S-homomorphism $\alpha : M_S \longrightarrow M_S$ which is extension of α(this means $\alpha \circ i = j \circ \alpha$). Now, define $\sigma(= \alpha^{-1}\pi\alpha): M_S \longrightarrow N$. If $n \in N$, write $\alpha(n) = k \in K$, hence $\sigma n = \alpha^{-1}(\pi\alpha(n)) \in N$, then $\sigma n =$

$\alpha^{-1}(\pi\alpha(n)) = \alpha^{-1}(\pi\alpha(n)) = \alpha^{-1}(\pi(k)) = \alpha^{-1}(k) = \alpha^{-1}(\alpha(n)) = n$ .Thus, $\sigma n = n$ and inclusion split, since $\sigma \circ i = I_N$.

Recall that an S-act M_S is called principally self-generator if every $x \in M_S$, there is an S-homomorphism $f: M_S \longrightarrow xS$ such that $x = f(x_1)$ for $x_1 \in M_S$[24].

Lemma (2.1.14): Let M_S be principally a self-generator (Abdul-Kareem, and Ahmed, 2022). Then, every principal subact is of the form mS, where $\gamma_S(m_0) \subseteq \gamma_S(m)$ and $M_S = m_0S$.

Proposition (2.1.15): Let M_S be a principal act, which is a principal self-generator, and let T =End(M). The following conditions are equivalent:

1. M_S is a small pseudo principally injective;

2. $S_{(\alpha,m)} = B_{\alpha m}\,\alpha \cup \ell_T\,(mS \times mS)$ for all $\alpha \in T$ and all $m \in M_S$;

3. If $A_{\alpha m} = A_{\beta m}$, then $\beta \in B_{\alpha m}\,\alpha \cup \ell_T\,(mS \times mS)$

Proof: (1→2) By proposition (2.1.10).

(2→3) Let $A_{\alpha m} = A_{\beta m}$, then $\gamma_S(\alpha m) = \gamma_S(\beta m)$. Let $(x, y) \in \ker\alpha$, so $\alpha(x) = \alpha(y)$ where $x, y \in M_S = mS$. Let $x = ms_1$ and $y = ms_2$, then $\alpha(m)\, s_1 = \alpha(m)\, s_2$, so $s_1, s_2 \in \gamma_S(\alpha\, m) = \gamma_S(\beta m)$. This implies $\beta(m)s_1 = \beta(m)s_2$ and then $\beta(ms_1) = \beta(ms_2)$, this means $\beta(x) = \beta(y)$ and $(x, y) \in \ker\beta$. Thus, $\ker\alpha \subseteq \ker\beta$. For the other direction, let $(x, y) \in \ker\beta$, so $\beta(x) = \beta(y)$ since $x, y \in M_S = mS$. Let $x = ms_1$ and $y = ms_2$. Thus $\beta(m)s_1 = \beta(m)s_2$ and then $s_1, s_2 \in \gamma_S(\beta m) = \gamma_S(\alpha\, m)$. This implies $\alpha(m)\, s_1 = \alpha(m)\, s_2$, then $\alpha(ms_1) = \alpha(m\, s_2)$, so $\alpha(x) = \alpha(y)$ which implies $(x, y) \in \ker\alpha$, thus $\ker\alpha = \ker\beta$. So, $\ker\beta \cap (mS \times mS) = \ker\alpha \cap (mS \times mS)$. which implies $S_{(\alpha,m)} = S_{(\beta,m)}$, so by (2), we have $B_{\alpha m}\,\alpha \cup \ell_T\,(mS \times mS) = B_{\beta m}\,\beta \cup \ell_T\,(mS \times mS)$. Since $1_M \in B_{\beta(m)}$. This means $\beta = 1_M \bullet \beta \in B_{\beta m}\,\beta$, so $\beta \in B_{\beta m}\,\beta \cup \ell_T\,(mS \times mS) = B_{\alpha m}\,\alpha \cup \ell_T\,(mS \times mS)$, this implies $\beta \in B_{\alpha m}\,\alpha \cup \ell_T\,(mS \times mS)$. Also, $\alpha \in B_{\beta m}\,\beta \cup \ell_T\,(mS \times mS)$.

(3→1) Assume that $f: mS \longrightarrow M_S$ be an S-homomorphism. Since M_S is principal, so there exists $m_0 \in M_S$ such that $M_S = m_0S$ and $\alpha: M_S \longrightarrow mS$ with $\alpha(m_0) = m$, where $\gamma_S(m_0) \subseteq \gamma_S(m)$. Again since M_S is principal self-generator, so there exists $\beta: M_S \longrightarrow f(m)S$ such that $f(m) = \beta(m_0)$,where $M_S = m_0S$... (1) Since f is S-monomorphism, so $\gamma_S(f\,m) = \gamma_S(m)$. In fact, since, if $s, t \in \gamma_S(f\,m)$, so $f(ms) = f(mt)$. Also, since f is monomorphism, so $ms = mt$ which implies $s, t \in \gamma_S(m)$ and then $\gamma_S(f(m)) \subseteq \gamma_S(m)$. For the other direction, let $s, t \in \gamma_S(m)$, so $ms = mt$. Since f is well-defined, so $f(ms) = f(mt)$. Thus, $f(m)s = f(m)t$ which implies $s, t \in \gamma_S(f(m))$ and then $\gamma_S(m) \subseteq \gamma_S(f(m))$. Thus, $\gamma_S(f(m)) = \gamma_S(m)$. This implies $\gamma_s(\beta(m_0)) = \gamma_s(\alpha(m_0))$. This means $\ker\alpha = \ker\beta$. In fact, for $(x, y) \in \ker\alpha$, this implies $\alpha(x) = \alpha(y)$ where where $x, y \in M_S = m_0S$. Let $x = m_0s_1$, and $y = m_0s_2$, then $\alpha(m_0s_1) = \alpha(m_0s_2)$ which implies $\alpha(m_0)s_1 = \alpha(m_0)s_2$, so $s_1, s_2 \in \gamma_s(\beta(m_0)) = \gamma_s(\alpha(m_0))$ by the proof. This implies $\beta(m_0)s_1 = \beta(m_0)s_2$ and then $\beta(m_0s_1) = \beta(m_0s_2)$, this means $\beta(x) = \beta(y)$ and $(x, y) \in \ker\beta$. Thus $\ker\alpha \subseteq \ker\beta$. Similarly for other direction, thus $\ker\alpha = \ker\beta$. So, $\ker\alpha \cap (m_0S \times m_0\,S) =$

$\ker\beta \cap (m_0S \times m_0 S)$ which implies $S_{(\alpha,m_0)} = S_{(\beta,m_0)}$ and $A_{\alpha m_0} = A_{\beta m_0}$, so by (3) we have $\beta \in B_{\alpha m_0}\alpha \cup \ell_T(m_0S \times m_0S)$. Thus, either $\beta \in B_{\alpha m_0}\alpha$ or $\beta \in \ell_T(m_0S \times m_0S)$. If $\beta \in B_{\alpha m_0}\alpha$, then there exists S-homomorphism $\varphi \in B_{\alpha m_0}$ which implies $\varphi \in T$ and $\beta = \varphi\alpha$. Thus, $\varphi(m) = \varphi(\alpha(m_0)) = \beta(m_0)$ and by (1) $\beta(m_0) = f(m)$, so $\varphi|_{mS} = f$, so M_S is small pseudo principally injective act. If $\beta \in \ell_T(m_0S \times m_0S)$, so $\beta \in \ell_T(M_S \times M_S)$ which implies $\beta \in T$ and $\forall(x,y) \in M_S \times M_S$, we have $\beta(x) = \beta(y), \forall(x,y) \in M_S$. This implies $\ker\beta = M_S \times M_S$ and then $\beta = 0$ which implies f = 0 and this is a contradiction.

The next theorem represents a generalization of theorem (2.5) in (Wongwai and Sthityanak,2012).

Theorem (2.1.16): Let M_S be a right S-act. If every small and principal subact of M_S is projective, then every factor act of a small PPM-injective act is small PPM-injective.

Proof: Let A be SPPM- injective act, mS be small subact of M_S. Let $\boldsymbol{\alpha}: \mathbf{mS} \longrightarrow \mathbf{A}/\boldsymbol{\rho}$ be a monomorphism where $\boldsymbol{\rho}$ is a congruence on A. Then by assumption where A is projective, so there exists S-homomorphism $\bar{\boldsymbol{\alpha}}: \mathbf{mS} \longrightarrow \mathbf{A}$ such that $\boldsymbol{\alpha} = \boldsymbol{\pi} \circ \bar{\boldsymbol{\alpha}}$ where $\boldsymbol{\pi}$ is the natural epimorphism $\boldsymbol{\pi}: \mathbf{A} \longrightarrow \mathbf{A}/\boldsymbol{\rho}$. I t is easy to check that $\bar{\boldsymbol{\alpha}}$ is monomorphism, for that let $\mathbf{x_1}, \mathbf{x_2} \in \mathbf{mS}$, if $\bar{\boldsymbol{\alpha}}(\mathbf{x_1}) = \bar{\boldsymbol{\alpha}}(\mathbf{x_2})$, then $\boldsymbol{\pi}\bar{\boldsymbol{\alpha}}(\mathbf{x_1}) = \boldsymbol{\pi}\bar{\boldsymbol{\alpha}}(\mathbf{x_2})$ which implies that $\boldsymbol{\alpha}(\mathbf{x_1}) = \boldsymbol{\alpha}(\mathbf{x_2})$. Since $\boldsymbol{\alpha}$ is monomorphism, so $\mathbf{x_1} = \mathbf{x_2}$ and this means that $\bar{\boldsymbol{\alpha}}$ is monomorphism. Since A is SPPM-injective, so there exists $\boldsymbol{\beta}: \mathbf{M_S} \longrightarrow \mathbf{A}$. Then $\boldsymbol{\pi\beta}$ is extension of $\boldsymbol{\alpha}$ to M_S and figure (2) illustrating that:

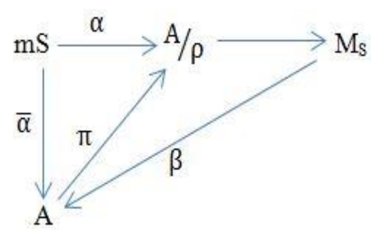

Figure 2. Clarifies that A/ρ is small PPM- injective acts

Proposition (2.1.17): Let M_S be a principal, small pseudo-PQ-injective act. If $\boldsymbol{\gamma_S}(\boldsymbol{\alpha}\mathbf{m}) = \boldsymbol{\gamma_S}(\boldsymbol{\beta}\mathbf{m})$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{T}$ with $\boldsymbol{\alpha}(\mathbf{M})$ is small in M_S then $\mathbf{T}\boldsymbol{\beta} \subseteq \mathbf{T}\boldsymbol{\alpha}$.

Proof: Let $\boldsymbol{\gamma_S}(\boldsymbol{\alpha}\mathbf{m}) = \boldsymbol{\gamma_S}(\boldsymbol{\beta}\mathbf{m})$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{T}$ with $\boldsymbol{\alpha}(\mathbf{M})$is small in M_S. Define $\boldsymbol{\varphi}: \boldsymbol{\alpha}(\mathbf{M}) \longrightarrow \mathbf{M_S}$ by $\boldsymbol{\varphi\alpha}(\mathbf{ms}) = \boldsymbol{\beta}(\mathbf{ms})$ for every $\mathbf{m} \in \mathbf{M_S}$ and $\mathbf{s} \in \mathbf{S}$. It is easy to check that $\boldsymbol{\varphi}$ is monomorphism. For this, let $\boldsymbol{\varphi}(\boldsymbol{\alpha}(\mathbf{m_1s})) = \boldsymbol{\varphi}(\boldsymbol{\alpha}(\mathbf{m_2s}))$. This implies to $\boldsymbol{\beta}(\mathbf{m_1s}) = \boldsymbol{\beta}(\mathbf{m_2s})$. Since $\boldsymbol{\gamma_S}(\boldsymbol{\alpha}\mathbf{m}) = \boldsymbol{\gamma_S}(\boldsymbol{\beta}\mathbf{m})$, so $\boldsymbol{\alpha}(\mathbf{m_1s}) = \boldsymbol{\alpha}(\mathbf{m_2s})$ and this means that $\boldsymbol{\varphi}$ is monomorphism. Since $\boldsymbol{\alpha}(\mathbf{M})$is small and principal subact of M_S and M_S is small pseudo-PQ-injective act, so there exists $\bar{\boldsymbol{\varphi}}$ which is extension of $\boldsymbol{\varphi}$. Then $\boldsymbol{\beta} = \boldsymbol{\varphi\alpha} = \bar{\boldsymbol{\varphi}}\boldsymbol{\alpha} \in \mathbf{T}\boldsymbol{\alpha}$ and so $\mathbf{T}\boldsymbol{\beta} \subseteq \mathbf{T}\boldsymbol{\alpha}$.

By similar way, one can prove the following proposition:

Proposition (2.1.18): Let M_S be small pseudo-PQ-injective act. If $\gamma_S(m) = \gamma_S(n)$, where $m, n \in M_S$ with mS is small in M_S, then $Tn \subseteq Tm$.

Proof: Let $\gamma_S(m) = \gamma_S(n)$, where $m, n \in M_S$ with mS is small in M_S. Define $\varphi: mS \longrightarrow M_S$ by $\varphi(ms) = ns$ for every $\in S$. It is easy to check that φ is S-monomorphism. For this, let $\varphi(m_1) = \varphi(m_2)$. This implies to $n_1 s = n_2 s$. Since $\gamma_S(m) = \gamma_S(n)$, so $m_1 s = m_2 s$ and this means that φ is S-monomorphism. Since mS is small and principal subact of M_S and M_S is SPPQ - injective-act, so there exists $\overline{\varphi}$ which is extension of φ. Then $n = \varphi(m) = \overline{\varphi}(m) \in Tm$ and so $Tn \subseteq Tm$.

Proposition (2.1.19): Let M_S be small pseudo-PQ-injective act $m \in M_S$, $t \in T$. and

1. If mS is a simple and small right S-act, then Tm is a simple left T-act.
2. If $\alpha(M)$ is a simple and small right S-act, then $T\alpha$ is a simple left T-act.

Proof: 1. Let $\Theta \neq \alpha m \in Tm$.Then $\alpha: mS \longrightarrow \alpha(mS)$ is an S-isomorphism by hypothesis, so let $\beta: \alpha(mS) \longrightarrow mS$ be the inverse of α. If $\overline{\beta} \in T$ extends β, then $m = \beta(\alpha(m)) = \overline{\beta}(\alpha(m)) \in T\alpha m$. This implies to $Tm = T\alpha m$

2. By the similar proof of (1).

Theorem (2.1.20): Let M_S be a small pseudo-PQ-injective act and torsion free act over cancellative monoid. Let m,n $\in M_S$ and mS is small subact in M_S:
1. If mS embeds in nS, then Tm is an image of Tn.
2. If nS is an image of mS, then Tn embeds in Tm
3. If mS ≅ nS, then Tm ≅ Tn .

Proof: (1) Let α: mS ⟶nS be S-monomorphism, so α(m) ∈ nS, then there exists s ∈ S such that α(m) = ns. Let i_1: mS⟶M_S and i_2: nS⟶M_S be the inclusion maps. Since M_S is small pseudo-PQ-injective, so there exists an S-homomorphism $\overline{\alpha}$: M_S⟶M_S such that $i_2\alpha = \overline{\alpha} i_1$ and figure (3) below explaining that.

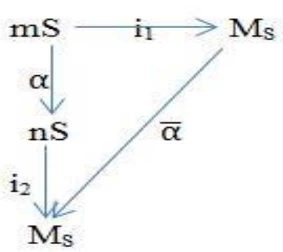

Figure 3. Explains that MS is small pseudo-PQ-injective act

Let β: Tn ⟶Tm defined by $\beta(\sigma(n)) = \sigma(\overline{\alpha}(m))$ for every σ ∈ T. Since β(σ(n)) = σα(m) ∈ σ(nS). So, for each σn ∈ Tn, f ∈ T we have $\beta(f(\sigma n)) = \beta(f\sigma)(n) = (f\sigma)\overline{\alpha}(m) = f(\sigma(\overline{\alpha}(m))) = f\beta(\sigma n)$. Thus, β is T-homomorphism. If $\sigma_1 n = \sigma_2 n$, where $\sigma_1, \sigma_2 \in T$, then $\sigma_1 n s_1 = \sigma_2 n s_1$ such that $s_1 \in S$. This implies that

$(\sigma_1,\sigma_2) \in \gamma_s(ns_1)$ and then $(\sigma_1,\sigma_2) \in \gamma_s(\overline{\alpha}m)$. Thus $\sigma_1(\overline{\alpha}m) = \sigma_2(\overline{\alpha}m)$ and since $\overline{\alpha}(m) = (\overline{\alpha}i_1)(m) = i_2\alpha(m) = \alpha(m)$. Therefore, $\beta(\sigma_1 n) = (\sigma_2 n)$, and so β is well-defined. We claim that $\gamma_s(\overline{\alpha}m) \subset \gamma_s(m)$, let $(s,t) \in \gamma_s(\overline{\alpha}m)$ which implies that $\overline{\alpha}(ms)=\overline{\alpha}(mt)$. This implies that $\alpha(ms)=\alpha(mt)$. Since α is monomorphism, so ms = mt, then $(s,t) \in \gamma_s(m)$. Thus, by proposition (2.18), we have $Tm \subset T\overline{\alpha}m$. For $\beta m \in T\overline{\alpha}m$, so there exists $\sigma \in T$ such that $\beta m = \sigma\overline{\alpha}(m) = \beta(\sigma n)$. Thus β is T-epimorphism.

(2) As in (1), let $\alpha: mS \rightarrow nS$ be S-epimorphism. Put $\alpha(ms) = n$, where $s \in S$. Since M_S is small pseudo-PQ-injective, so α can be extended to $\overline{\alpha} : M_S \rightarrow M_S$ such that $i_2\alpha = \overline{\alpha}i_1$. Define $\beta: Tn \rightarrow Tm$ by $\beta(\sigma(n)) = \sigma(\overline{\alpha}(ms))$ for every $\sigma \in T$ and $s \in S$. From (1) β is T-homomorphism. Since α is epimorphism, so there exists $s \in S$ such that n = (ms). Let $(\sigma_1 n,\sigma_2 n) \in \ker\beta$, then $\beta(\sigma_1 n) = \beta(\sigma_2 n)$ which implies that $\beta(\sigma_1(\alpha(ms)) = \beta(\sigma_2(\alpha(ms))$, then $\sigma_1(\overline{\alpha}(ms)) = \sigma_2(\overline{\alpha}(ms))$. Then, $\sigma_1(\alpha(ms)) = \sigma_2(\alpha(ms))$. Thus $\sigma_1 n = \sigma_2 n$ and β is T-monomorphism.

(3) By (1) and (2), if $\alpha: mS \longrightarrow nS$ is S-isomorphism, then $\beta: Tn \longrightarrow Tm$ is T-isomorphism

Relationship between Small Pseudo Principally Quasi-Injective S-Acts with Other Classes of Injectivity

It is well known that each Small PQ-injective act is a small pseudo-PQ-injective, but the converse is generally not true. To show under which conditions the converse is true, we need the following concepts, propositions, and lemmas. Mehdi and Hiba (2017) define fully small stable modules and extend this definition to acts and define it as follows:

Definition (2.2.1): Hiba and Mohanad (2021) stated that a small subact N of an S-act M_S is called stable if $f(N) \subseteq N$ for each S-homomorphism $f : N \longrightarrow M_S$. An S-act M_S is called fully small stable if each small subact of M_S is stable.

Recall that a subact N of an S-act M_S is called stable if $f(N) \subseteq N$ for each S-homomorphism

$f: N \longrightarrow M_S$. An S-act M_S is called fully stable if each subact of M_S is stable (Hiba, 2014).

It is clear that every fully stable act is a fully small stable act, but the converse is not true generally. For example, Z as Z-act is fully small and stable since the only small subact is the one element (Θ). But Z as Z-act is not fully stable since there exists a homomorphism $f : 2Z \longrightarrow Z$ defined by $f(2a) = 3a$ for each $a \in Z$. It is easy to check that $f(2Z) \nsubseteq 2Z$.

Besides, Z-act Q with usual multiplication is not a fully small stable act. Since it is well-known that Q has no maximal subacts

Recall that an S-act M_S is multiplication if each subact of M_S is of the form MI, for some right ideal I of S. This is equivalent to saying that every principal subact is of this form (Mohammad and Majid, 2014). Abbas and Mustafa (2015) define pseudo stable subact as follows: a subact N of M_S is called pseudo stable if $f(N) \subseteq N$ for each S-monomorphism $f: N \longrightarrow M_S$. An S-act M_S is called fully pseudo-stable in case each subact of MS is pseudo-stable. A monoid S is fully pseudo-stable if it is a fully pseudo-stable S-act.

The above definition motivated us to generalize it as follows:

Definition (2.2.2): A right S-act M_S is called fully small pseudo stable act if every small subact A of M_S is pseudo stable.

Proposition (2.2.3): Let M_S be multiplication S-act. Then, M_S is fully small pseudo-stable if and only if MS is a small pseudo-PQ-injective S-act.

Proof: Let mS be a small and principal subact of an S-act M_S and $\alpha : mS \longrightarrow M_S$ be an S-monomorphism, where $m \in M_S$. Then, since M_S is a small pseudo-PQ-injective, so α extends to an S-homomorphism $\beta : M_S \longrightarrow M_S$. Since M_S is a multiplication act, so there is an ideal I of S such that mS = MI. Hence, $\alpha(mS) = \beta(mS) = \beta(MI) = \beta(M)\, I \subseteq MI = mS$. Thus, M_S is fully small pseudo stable.

Proposition (2.2.4): The following are equivalent to an S-act M_S.

1. M_S is fully small pseudo-stable.

2. Every subact of M_S is fully small pseudo-stable.

3. Every 2-generated subact of M_S is fully small pseudo-stable.

4. If N, K are subacts of M_S and $N \cong K$, then N = K.

5. $\gamma_S(x) = \gamma_S(y)$ implies that xS = yS for some x, y in M_S.

Proof: (1⟹2) and (2 ⟹ 3) are obvious.

(3⟹1) Suppose N is a small subact of M_S and $\alpha : N \longrightarrow M$ is an S-monomorphism. Let n be an element of N and let $K = nS \cup \alpha(n)S$. Let $\beta = \alpha|_{nS} : nS \longrightarrow M$. Then, clearly, $\alpha(n) = \beta(n)$. By assumption, K is fully small pseudo-stable and so $\alpha(n) \in nS$. It follows that M_S is fully small pseudo-stable.

(1⟹4) If N, K are two subacts of M_S and $\alpha : N \longrightarrow K$ is an S-monomorphism, then $K = \alpha(N) \subseteq N$.Since $\alpha^{-1}: K \longrightarrow N$ is also S-isomorphism, then $N = \alpha^{-1}(K) \subseteq K$. Hence N = K.

(4⟹5) Suppose $\gamma_S(x) = \gamma_S(y)$ for some $x, y \in M_S$. Define $\alpha : xS \longrightarrow yS$ by $\alpha(xs) = ys$ for every $s \in S$. Clearly, α is a well- defined isomorphism and so xS = yS.

(4⟹1) Let N be any small subact of M_S and $\alpha : N \longrightarrow M$ is an S-monomorphism. Let $n \in N$, then $\gamma_S(n) = \gamma_S(\alpha(n))$ and hence $\alpha(n) \in \alpha(n)S = nS \subseteq N$. Consequently, N is small pseudo-stable.

Lemma (2.2.5): [28] Let M_S be an S-act where S is a commutative monoid and ρ be a congruence on S. Then $\ell_M(\rho) \cong Hom_S(S/\rho, M_S)$

The next proposition represents a generalization of proposition (2.5) (Abbas, and Mustafa, 2015):

Proposition (2.2.6): An S-act M_S is fully small if and only if M_S is tiny pseudo stable and xS≅Hom (xS, M_S) for each x in M_S.

Proof: Let M_S is a fully small stable S-act, then $\ell_M(\gamma_S(a)) = aS$ for each $a \in M_S$. By lemma (3.4) $aS = \ell_M(\gamma_S(a)) \cong Hom(S/\gamma_S(a), M_S) \cong Hom(aS, M_S)$. Conversely, for each $x \in M_S$, we have

$xS \cong Hom(xS, M_S) \cong Hom(S/\gamma_S(a), M_S) \cong \ell_M(\gamma_S(a))$. Then by proposition (2.2.3) implies that $xS \cong \ell_M(\gamma_S(a))$. Thereby, M_S is fully small stable.

The next proposition represents a generalization of proposition (2.22) (Shaymaa, 2018):

Proposition (2.2.7): Let S be a commutative monoid and M_S be a multiplication S-act. Then M_S is fully small stable if and only if M_S is a Small PQ-injective act.

Proof: ⟹)It is clear.

⟸)Let $\alpha : mS \longrightarrow M_S$ be S-homomorphism, where $m \in M_S$. Then, since M_S is Small PQ-injective act, so α extends to S- homomorphism $\beta : M_S \rightarrow M_S$. Now, an ideal I of S exists, such that mS = MI. Hence $\alpha(mS) = \beta(MI) = \beta(M)I \subseteq MI = mS$

Now, since every cyclic (principal) act is multiplication (For, if N is a subact of a cyclic S-act M_S =mS and $x \in N$ then, $x \in M_S$ so x = ms where s belongs to the ideal of S and m belongs to M_S. Hence, N = MI). Then, we have the following corollary:

Corollary (2.2.8): A cyclic (principal) S-act M_S is fully small stable if and only if M_S is Small PQ-injective.

The next proposition gives conditions on Small Pseudo-PQ-injective acts to be Small PQ-injective.

Proposition (2.2.9): Let M_S be multiplication S-act, where S is a commutative monoid and $xS \cong Hom(xS, M_S)$ for each x in M_S. If M_S is a small pseudo-PQ-injective act, then MS is a Small PQ-injective.

Proof: Assume that M_S is a small pseudo-PQ-injective act. Since M_S is a multiplication act, M_S is a tiny pseudo stable by proposition (2.2.3). Since $xS \cong Hom(xS, M_S)$, so by proposition (2.2.6), M_S is a tiny stable act. Again, since M_S is a multiplication act, so by proposition (2.2.7), MS is a Small PQ-injective act.

At the same time, we can give other conditions to versus small pseudo-PQ-injective S-acts with Small PQ-injective, but we need the following concept:

Definition (2.2.10): An S-act M_S is called cog-reversible if each congruence ρ on M_S with $\rho \neq I_M$ is large on M_S (Shaymaa, 2015).

For example, Z-acts Z and Q are cog-reversible. As every congruence ρ on Z_Z (and Q_Z) with $\rho \neq I_Z$ (and $\rho \neq I_Q$) is large on Z_Z (and Q_Z).

Proposition (2.2.11): Let M_S be a cog-reversible nonsingular S-act with $\ell_M(s) = \Theta, \forall s \in S$ .If M_S is small pseudo-PQ-injective, then M_S is Small PQ-injective.

Proof: Let N be the small and principal subact of S-act M_S and f be S-homomorphism from N into M_S. If f is S-monomorphism, then there is nothing to prove. So, assume f is not S-monomorphism. Then, using the proof of the theorem (3.2.17) (Shaymaa, 2015). we get the required. This means that M_S is Small PQ-injective S-act.

Conclusion

In this paper, we introduced a novel concept called Small pseudo-PQ-injective acts. We achieved several intriguing findings, like proposition (2.2.6), which was a generalization to the proposition (2.5) in (Abbas & Mustafa, 2015), proposition (2.2.7) which was a generalization to the proposition (2.22) in (Shaymaa, 2018a), theorem (2.1.16) which was a generalization to the theorem (2.5) in (Wongwai, and Sthityanak, 2012) and definition(2.2.2) of the tiny pseudo stable act which was a generalization to the definition of fully pseudo stable act presented by Abbas and Mustafa in (Abbas, and Mustafa, 2015). In addition, various new characterizations are clarified for that concept, such as remarks and examples (2.1.2) (3), lemma (2.1.3), and proposition (2.1.4). We also discovered the link between the classes of small pseudo-PQ-injective acts, the classes of injectivity, and the conditions for coinciding between these classes, like proposition (2.2.9).

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